

# THE BERGMAN ANALYTIC CONTENT OF PLANAR DOMAINS

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ABSTRACT. Given a planar domain  $\Omega$ , the *Bergman analytic content* measures the  $L^2(\Omega)$ -distance between  $\bar{z}$  and the Bergman space  $A^2(\Omega)$ . We compute the Bergman analytic content of simply-connected quadrature domains with quadrature formula supported at one point, and we also determine the function  $f \in A^2(\Omega)$  that best approximates  $\bar{z}$ . We show that, for simply-connected domains, the square of Bergman analytic content is equivalent to *torsional rigidity* from classical elasticity theory, while for multiply-connected domains these two domain constants are not equivalent in general.

## 1. INTRODUCTION

Recall that, for a bounded domain  $\Omega$  in  $\mathbb{C}$ , the Bergman space  $A^2(\Omega)$  is the Hilbert space of functions holomorphic in  $\Omega$  that satisfy  $\int_{\Omega} |f(z)|^2 dA(z) < \infty$ . Extending previous studies on the approximation of  $\bar{z}$  in analytic function spaces, in [6] the authors introduced the notion of *Bergman analytic content*, defined as the  $L^2(\Omega)$ -distance between  $\bar{z}$  and the space  $A^2(\Omega)$ . They showed that the best approximation to  $\bar{z}$  is 0 if and only if  $\Omega$  is a disk, and that the best approximation is  $\frac{c}{z}$  if and only if  $\Omega$  is an annulus centered at the origin.

D. Khavinson and the first author in [5] reduced the problem of finding the best approximation to that of solving a Dirichlet problem in  $\Omega$  with boundary data  $|z|^2$ . Using this fact, we will determine the best approximation and Bergman analytic content when  $\Omega$  is a simply-connected, one-point quadrature domain.

Recall that a bounded domain  $\Omega \subset \mathbb{C}$  is called a *quadrature domain* if it admits a formula expressing the area integral of any test function  $g \in A^2(\Omega)$  as a finite sum of weighted point evaluations of  $g$  and its derivatives:

$$(1.1) \quad \int_D g(z) dA(z) = \sum_{m=1}^N \sum_{k=0}^{n_m} a_{m,k} g^{(k)}(z_m),$$

where the points  $z_m \in \Omega$  and constants  $a_{m,k}$  are each independent of  $g$ . This class of domains is  $C^\infty$ -dense in the space of domains having a  $C^\infty$ -smooth boundary [2, Thm. 1.7], and the restricted class of quadrature domains for which  $N = 1$  in (1.1) has the same density property. When  $\Omega$  is a simply-connected quadrature domain with  $N = 1$ , the conformal mapping  $\phi : \mathbb{D} \rightarrow \Omega$  is a polynomial, and by making a translation we may assume that the quadrature formula is supported at  $\phi(0) = 0$ .

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{C}$  be a simply-connected quadrature domain with quadrature formula supported at a single point (say the origin), and let  $\phi : \mathbb{D} \rightarrow \Omega$  be the (polynomial) conformal map from the unit disk*

$$\phi(z) = \sum_{k=1}^n a_k z^k.$$

*Then the Bergman analytic content  $\lambda_{A^2}(\Omega)$  of  $\Omega$  is given by:*

$$(1.2) \quad \lambda_{A^2}(\Omega) = \pi^{1/2} \left[ \sum_{m=1}^{2n-1} \frac{|c_m|^2}{m+1} - \sum_{k=1}^{n-1} k \left| \sum_{j=1}^{n-k} a_{k+j} \overline{a_j} \right|^2 \right]^{1/2},$$

where

$$c_m := \sum_{k+j=m+1} k a_k a_j \quad 1 \leq k, j \leq n.$$

Moreover, the best approximation  $f$  to  $\bar{z}$  is the derivative  $f = F'$  of  $F = P \circ \phi^{-1}$ , where

$$P(\zeta) = \frac{1}{2} \sum_{k=1}^n |a_k|^2 \zeta^k + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{k+j} \overline{a_j} \zeta^k.$$

In [5], the authors characterized domains for which the best approximation to  $\bar{z}$  is a polynomial, showing in particular that the only quadrature domain with this property is a disk. On the other hand, Theorem 1.1 reveals that for a large class of quadrature domains the best approximation  $f$  has a primitive  $F$  that becomes a polynomial in the right coordinate system.

Let  $\Omega$  be a domain in the plane, bounded by finitely many Jordan curves  $\Gamma_0, \dots, \Gamma_n$ , with  $\Gamma_0$  the outer boundary curve. Then the *torsional rigidity* of  $\Omega$  equals

$$(1.3) \quad \rho(\Omega) := \int_{\Omega} |\nabla \nu|^2 dA,$$

where  $\nu$  solves the Dirichlet problem

$$\begin{cases} \Delta \nu = -2 & \text{in } \Omega \\ \nu|_{\Gamma_0} = 0 \\ \nu|_{\Gamma_i} = c_i & i = 1, \dots, n \end{cases},$$

where the constants  $c_i$  are not known *a priori* but are determined from the conditions

$$\int_{\Gamma_i} \partial_n \nu ds = 2a_i, \quad i = 1, \dots, n,$$

where  $\partial_n$  denotes differentiation in the direction of the outward normal,  $ds$  is the arclength element, and  $a_i$  is the area enclosed by  $\Gamma_i$  (cf. [1, pp. 63-66]). The function  $\nu$  is referred to as the “stress function” in elasticity theory, and the torsional rigidity measures the resistance to twisting of a cylindrical beam with cross section  $\Omega$ .

In [5] the inequality

$$(1.4) \quad \sqrt{\rho(\Omega)} \leq \lambda_{A^2}(\Omega).$$

was shown to hold for simply-connected domains. We show that this is an equality for simply-connected domains.

**Theorem 1.2.** *Suppose  $\Omega$  is a bounded, simply-connected domain. Then  $\lambda_{A^2}(\Omega)^2 = \rho(\Omega)$ .*

Theorem 1.2 led us to notice that some of the methods and examples in the current paper overlap with classical studies of torsional rigidity [7, Ch. 22], and while the explicit formulas in Theorem 1.1 appear to be new, our proof based on conformal mapping is very similar to the procedure described in [7, Sec. 134]. The square of Bergman analytic content is not in general equivalent to torsional rigidity. This follows from explicit computations for doubly-connected domains such as the annulus (see Section 4).

As an extension of Theorem 1.1, it would be interesting to determine the best approximation to  $\bar{z}$  and the Bergman analytic content for general quadrature domains. Since the problem again reduces to solving a Dirichlet problem with data  $z\bar{z}$ , the procedure developed in [3] for solving the Dirichlet problem with rational (in  $z$  and  $\bar{z}$ ) data seems promising.

**Outline.** We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. We discuss examples in Section 4. In Section 5, we revisit the class of domains considered in [5] for which the best approximation to  $\bar{z}$  is a monomial.

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## 2. PROOF OF THEOREM 1.1

By the definition of Bergman analytic content, we have  $\lambda_{A^2}(\Omega) = \|\bar{z} - f\|_2$ , where  $f$  is the projection of  $\bar{z}$  onto  $A^2(\Omega)$ . By the Pythagorean theorem we then have that

$$\begin{aligned} \lambda_{A^2}(\Omega) &= \left( \int_{\Omega} |\bar{z}|^2 dA(z) - \int_{\Omega} |f(z)|^2 dA(z) \right)^{1/2} \\ &= \left( \int_{\mathbb{D}} |\bar{\phi}\phi'|^2 dA - \int_{\mathbb{D}} |f \circ \phi|^2 |\phi'|^2 dA \right)^{1/2}, \end{aligned}$$

where we have changed variables  $z = \phi(\zeta)$ ,  $dA(z) = |\phi'(\zeta)|^2 dA(\zeta)$ . The first term  $\int_{\mathbb{D}} |\bar{\phi}\phi'|^2 dA = \int_{\mathbb{D}} |\phi\phi'|^2 dA$  is simply the square of the Bergman norm of a polynomial  $\phi\phi'$ :

$$(2.1) \quad \int_{\mathbb{D}} |\bar{\phi}\phi'|^2 dA = \pi \sum_{m=1}^{2n-1} \frac{|c_m|^2}{m+1},$$

where

$$c_m := \sum_{k+j=m+1} k a_k a_j \quad 1 \leq k, j \leq n,$$

are the coefficients in the expansion of the product  $\phi \cdot \phi'$ .

In order to compute  $\int_{\Omega} |f(z)|^2 dA(z)$ , we first find  $f$  explicitly. By [5, Thm. 1],  $f = F'$ , where  $u = \operatorname{Re}(F)$  solves the Dirichlet problem

$$\begin{cases} \Delta u &= 0 \\ u|_{\partial\Omega} &= \frac{|z|^2}{2}. \end{cases}$$

Changing coordinates using the conformal map  $\phi$ , we obtain a harmonic function  $\tilde{u} = u \circ \phi$  that solves the following Dirichlet problem in the unit disk:

$$\begin{cases} \Delta \tilde{u} &= 0 \\ \tilde{u}|_{\mathbb{T}} &= \frac{\phi\bar{\phi}}{2}. \end{cases}$$

Now, on  $\mathbb{T}$  we have that  $\phi\bar{\phi} = P(\zeta) + \overline{P(\zeta)}$ , where

$$P(\zeta) = \frac{1}{2} \sum_{k=1}^n |a_k|^2 + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{k+j} \bar{a}_j \zeta^k.$$

Since this is a harmonic polynomial, we have that  $\tilde{u}(\zeta) = \operatorname{Re}(P(\zeta))$ . Thus,  $F \circ \phi = P$ , and so by the chain rule  $(f \circ \phi)(\phi') = p$ , where

$$p(\zeta) = P'(\zeta) = \sum_{k=1}^{n-1} k \sum_{j=1}^{n-k} a_{k+j} \bar{a}_j \zeta^{k-1}.$$

Calculating the Bergman norm of this polynomial, we find that

$$(2.2) \quad \int_{\Omega} |f(z)|^2 dA(z) = \int_{\mathbb{D}} |f \circ \phi|^2 |\phi'|^2 dA = \sum_{k=1}^{n-1} k \left| \sum_{j=1}^{n-k} a_{k+j} \bar{a}_j \right|^2.$$

Combining (2.1) and (2.2), the result follows.

### 3. PROOF OF THEOREM 1.2

Recall that if  $\Omega$  is a simply connected domain, the torsional rigidity  $\rho = \rho(\Omega)$  is given by equation (1.3)

$$(3.1) \quad \rho = \int_{\Omega} |\nabla \nu|^2 dA,$$

where  $\nu$  is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta \nu &= -2 \\ \nu|_{\partial\Omega} &= 0 \end{cases}.$$

Consider the function  $u(z) := \nu(z) + \frac{|z|^2}{2}$ . Then  $u$  solves the Dirichlet problem stated in the proof of Theorem 1.1:

$$\begin{cases} \Delta u &= 0 \\ u|_{\partial\Omega} &= \frac{|z|^2}{2} \end{cases}.$$

Thus,  $u = \operatorname{Re}(F)$ , where  $f = F'$  is the best approximation to  $\bar{z}$ .

Letting  $\nu$  denote the torsion function, we have:

$$\begin{aligned} \rho(\Omega) &= \int_{\Omega} |\nabla \nu|^2 dA \\ &= \int_{\Omega} |2\partial_z \nu|^2 dA \\ &= \int_{\Omega} \left| 2\partial_z u - 2\partial_z \frac{|z|^2}{2} \right|^2 dA \\ &= \int_{\Omega} |F' - \bar{z}|^2 dA \\ &= \int_{\Omega} |\bar{z} - f|^2 dA \\ &= \int_{\Omega} |z|^2 - |f|^2 dA \\ &= \lambda_{A^2}(\Omega)^2, \end{aligned}$$

and this completes the proof.

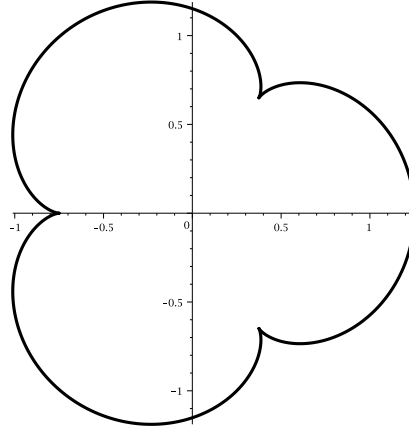
#### 4. EXAMPLES

**4.1. Epicycloids.** Let us consider the one-parameter family of domains  $\Omega$  with conformal map  $\phi : \mathbb{D} \rightarrow \Omega$ , given by  $\phi(z) = z + az^n$ , with  $0 \leq a \leq 1/n$ .

Applying Theorem 1.1, we immediately obtain:

$$\lambda_{A^2}(\Omega) = \sqrt{\frac{\pi(1 + 4a^2 + na^4)}{2}}.$$

When  $a = 1/n$  the domain develops cusps (the case  $n = 4$  is plotted in Figure 4.1). The case  $n = 2$  and  $a = 1/2$  is a cardioid (cf. [8, Sec. 58]).

FIGURE 4.1. The epicycloid domain when  $n = 4$ ,  $a = 1/4$ .

**4.2. The annulus.** The following example shows that Theorem 1.2 does not hold in general for multiply-connected domains. Let  $\Omega = \{z : r < |z| < R\}$  be the annulus. The best approximation to  $\bar{z}$  in  $A^2(\Omega)$  is  $f(z) = \frac{C}{z}$ , where

$$C = \frac{R^2 - r^2}{2(\log R - \log r)}$$

(cf. [5] and [6]). Following the proof in Section 2, we have that

$$(4.1) \quad \lambda_{A^2}(\Omega)^2 = \int_{\Omega} |z|^2 - |f|^2 dA.$$

Integrating in polar coordinates we get that

$$\int_{\Omega} |z|^2 dA = \frac{\pi}{2}(R^4 - r^4),$$

and

$$\begin{aligned} \int_{\Omega} \left| \frac{C}{z} \right|^2 dA &= 2\pi C^2 \int_r^R \frac{1}{\rho} d\rho \\ &= \frac{\pi}{2} \frac{(R^2 - r^2)^2}{\log R - \log r}. \end{aligned}$$

Thus, we have that

$$\lambda_{A^2}(\Omega)^2 = \frac{\pi}{2} \left( (R^4 - r^4) - \frac{(R^2 - r^2)^2}{\log R - \log r} \right),$$

which is smaller than the torsional rigidity [1, p. 64] of  $\Omega$ :

$$\rho(\Omega) = \frac{\pi}{2} (R^4 - r^4).$$

So we find that neither Theorem 1.2 nor the inequality (1.4) hold for multiply-connected domains.

**4.3. The annular region bounded by a pair of confocal ellipses.** We consider the region  $G$  between two confocal ellipses that is the image of an annulus  $\Omega := \{z \in \mathbb{C} : r < |z| < R\}$  under a Joukowski map  $\phi(z) = z + \frac{1}{z}$ .

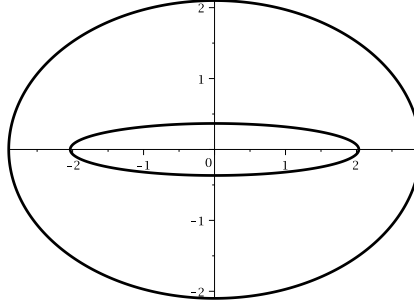


FIGURE 4.2. The annular region  $G$  when  $r = 1.2$ ,  $R = 2.5$ .

Following the proof in Section 2, the projection of  $\bar{z}$  to the Bergman space is given by  $f = F'$ , where  $u = \operatorname{Re}(F)$  solves the Dirichlet problem

$$\begin{cases} \Delta u &= 0 \\ u|_{\partial G} &= \frac{|z|^2}{2} \end{cases}.$$

The function  $\tilde{u} = u \circ \phi$  is harmonic and solves the following Dirichlet problem in the annulus  $\Omega := \{\zeta \in \mathbb{C} : r < |\zeta| < R\}$ :

$$\begin{cases} \Delta \tilde{u} &= 0 \\ \tilde{u}|_{\partial \Omega} &= \frac{\phi \bar{\phi}}{2} \end{cases}.$$

We make the ansatz

$$2\tilde{u}(\zeta) = A + B \log |\zeta| + C(\zeta^2 + \bar{\zeta}^2) + D \left( \frac{1}{\zeta^2} + \frac{1}{\bar{\zeta}^2} \right).$$

The boundary condition gives:

$$2\tilde{u}(\zeta) = |\zeta|^2 + \frac{1}{|\zeta|^2} + \frac{\zeta}{\bar{\zeta}} + \frac{\bar{\zeta}}{\zeta}, \quad \text{on } \partial \Omega.$$

Using polar coordinates to parameterize the two circular boundary components  $z = re^{i\theta}$  and  $z = Re^{i\theta}$ , we obtain two equations:

$$\begin{aligned} A + B \log r + 2 \left( Cr^2 + \frac{D}{r^2} \right) \cos(2\theta) &= r^2 + \frac{1}{r^2} + 2 \cos(2\theta), \\ A + B \log R + 2 \left( CR^2 + \frac{D}{R^2} \right) \cos(2\theta) &= R^2 + \frac{1}{R^2} + 2 \cos(2\theta), \end{aligned}$$

which implies the system of equations for  $A, B, C, D$

$$\begin{aligned} Cr^2 + \frac{D}{r^2} &= 1, \\ CR^2 + \frac{D}{R^2} &= 1, \\ A + B \log R &= R^2 + \frac{1}{R^2}, \\ A + B \log r &= r^2 + \frac{1}{r^2}. \end{aligned}$$

Solving this (linear in  $A, B, C, D$ ) system, we obtain:

$$\begin{aligned} A &= \frac{-\log r}{\log R - \log r} \left( R^2 + \frac{1}{R^2} \right) + \frac{\log R}{\log R - \log r} \left( r^2 + \frac{1}{r^2} \right), \\ B &= \frac{1}{\log R - \log r} \left( R^2 + \frac{1}{R^2} - r^2 - \frac{1}{r^2} \right), \\ C &= \frac{1}{R^2 + r^2}, \\ D &= \frac{r^2 R^2}{R^2 + r^2}. \end{aligned}$$

We have

$$(f \circ \phi)\phi' = \frac{B}{2\zeta} + C\zeta - \frac{D}{\zeta^3},$$

and thus the square of the Bergman norm of  $f$  is

$$\begin{aligned} \int_G |f(z)|^2 dA(z) &= \int_\Omega |f \circ \phi|^2 |\phi'|^2 dA \\ &= \int_\Omega \left| \frac{B}{2\zeta} + C\zeta - \frac{D}{\zeta^3} \right|^2 dA \\ &= \frac{\pi}{2} \left( B^2(\log R - \log r) + C^2(R^4 - r^4) + D^2 \left( \frac{1}{r^4} - \frac{1}{R^4} \right) \right). \end{aligned}$$



The square of the Bergman norm of  $\bar{z}$  is

$$\begin{aligned}
 \int_G |z|^2 dA(z) &= \int_\Omega |\phi\phi'|^2 dA \\
 &= \int_\Omega \left| \left( \zeta + \frac{1}{\zeta} \right) \left( 1 - \frac{1}{\zeta^2} \right) \right|^2 dA(\zeta) \\
 &= \int_\Omega \left| \zeta - \frac{1}{\zeta^3} \right|^2 dA(\zeta) \\
 &= \frac{\pi}{2} \left( R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4} \right).
 \end{aligned}$$

Thus,  $\lambda_{A^2}(G)^2 = \int_G |z|^2 dA(z) - \int_G |f(z)|^2 dA(z)$  is given by:

$$\frac{\pi}{2} \left( R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4} - \frac{1}{\log R - \log r} \left( R^2 + \frac{1}{R^2} - r^2 - \frac{1}{r^2} \right)^2 - 2 \frac{R^2 - r^2}{R^2 + r^2} \right).$$

## 5. DOMAINS FOR WHICH THE BEST APPROXIMATION IS A MONOMIAL

The domains defined by  $C\operatorname{Re}(z^n) - |z|^2 + 1 > 0$  represent an interesting class of examples (cf. [5]). These are the domains for which the best approximation to  $\bar{z}$  is a monomial, namely,  $\frac{Cn}{2}z^{n-1}$ . However, as indicated in Figure 5.1, there are values of  $C$  for which the set  $\{z : C\operatorname{Re}(z^n) - |z|^2 + 1 > 0\}$  does not have a bounded component, and  $\bar{z}$  is no longer in  $L^2(\Omega)$ . Here we address the question of what range of  $C$  leads to a bounded component.

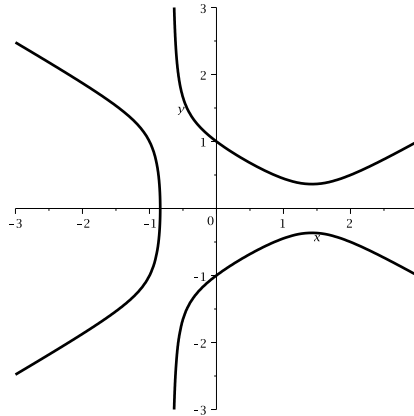


FIGURE 5.1. The region  $\{z : \frac{1}{2}\operatorname{Re}(z^3) - |z|^2 + 1 > 0\}$  does not have a bounded component.

**Proposition 5.1.** *The set  $\{z : C\operatorname{Re}(z^n) - |z|^2 + 1 > 0\}$  has a bounded component whenever*

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

*Proof.* Take  $z = re^{i\theta}$  and let  $f(r, \theta) := C \cos(n\theta)r^n - r^2 + 1$  be the defining function of the domain in polar coordinates. We will show that when

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}$$

we have  $f(R, \theta) \leq 0$  for all  $\theta$ , where  $R = (\frac{2}{nC})^{1/(n-2)}$ . Since the region  $\{z : C\operatorname{Re}(z^n) - |z|^2 + 1 > 0\}$  clearly contains the origin, this ensures that it has a component entirely contained in the disk  $|z| < R$ .

It is enough to show that  $f(R, 0) \leq 0$  since we have  $f(R, \theta) \leq f(R, 0)$ .

The function  $F(r) := f(r, 0) = Cr^n - r^2 + 1$ , has derivative  $F'(r) = Cnr^{n-1} - 2r$ , with a critical point at  $R = (\frac{2}{nC})^{1/(n-2)}$ , which by the first derivative test is a local minimum. Plugging this critical point into  $F$ , we find that

$$C \left( \frac{2}{nC} \right)^{n/(n-2)} - \left( \frac{2}{nC} \right)^{2/(n-2)} + 1 \leq 0$$

precisely when

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

□

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